

Omni-Lie Color Algebras and Lie Color 2-Algebras

Tao Zhang

Abstract Omni-Lie color algebras over an abelian group with a bicharacter are studied. The notions of 2-term color L_∞ -algebras and Lie color 2-algebras are introduced. It is proved that there is a one-to-one correspondence between Lie color 2-algebras and 2-term color L_∞ -algebras.

1 Introduction

The notion of omni-Lie algebra was introduced by Weinstein in [17] as a linearization of the Courant algebroid which was introduced by Liu, Xu and Weinstein in [9]. An omni-Lie algebra associated to a vector space V is the direct sum space $\mathfrak{g}(V) \oplus V$ together with the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and the skew-symmetric bracket operation $\llbracket \cdot, \cdot \rrbracket$ given by

$$\langle A + u, B + v \rangle = \frac{1}{2}(Av + Bu),$$

and

$$\llbracket A + u, B + v \rrbracket = [A, B] + \frac{1}{2}(Av - Bu).$$

The bracket $\llbracket \cdot, \cdot \rrbracket$ does not satisfy the Jacobi identity so that an omni-Lie algebra is not a Lie algebra. An omni-Lie algebra is actually a Lie 2-algebra since Roytenberg and Weinstein proved that every Courant algebroid gives rise to a Lie 2-algebra ([12]). Recently, omni-Lie algebras are studied from several aspects ([3], [6], [16]) and are generalized to omni-Lie algebroids and omni-Lie 2-algebras in [4, 5, 15]. The corresponding Dirac structures are also studied therein.

In this paper, we introduce the notion of omni-Lie color algebra, which is a color analogue of Weinstein's omni-Lie algebras. The nature of the omni-Lie color algebra remained unclear until we give the concept of 2-term

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color L_∞ -algebras. Certainly, we can construct a 2-term color L_∞ -algebras through omni-Lie color algebras. We also introduce an equivalent notion: Lie color 2-algebras.

The paper is organized as follows. In Section 2, we recall some facts about Lie color algebras and Leibniz color algebras. In Section 3, we define omni-Lie color algebra on $\mathfrak{gl}(V) \oplus V$ for a G -graded vector space V , where G is an abelian group. We study Dirac structures of omni-Lie color algebra in order to characterize all possible Lie color algebra structures on the G -graded vector space V .

In Section 4, we introduce the notion of 2-term color L_∞ -algebra. We prove that omni-Lie color algebras are 2-term color L_∞ -algebras. We also give examples different from omni-Lie color algebras, such as string Lie color 2-algebras and crossed modules of Lie color algebras. In Section 5, we introduce the notion of Lie color 2-algebra. We prove that there is a one-to-one correspondence between Lie color 2-algebras and 2-term color L_∞ -algebras.

2 Lie Color Algebras and Leibniz Color Algebras

We first recall some facts and definitions about Lie color algebras, basic reference is [14]. We work on a fixed field \mathbb{K} of characteristic 0.

Let G be abelian group. A G -graded vector space is a vector space with direct sum decomposition $V = \bigoplus_{\alpha \in G} V_\alpha$. An element of V is said to be homogeneous of degree α if it is an element of V_α . We denote the degree of a homogeneous element by $|x| := \alpha$ for $x \in V_\alpha$. A sub-vector space W of V is graded if $W = \bigoplus_{\alpha \in G} (W \cap V_\alpha)$.

A homomorphism $f : V \rightarrow W$ between two G -graded vector spaces V and W is a grade-preserving linear map: $f(V_\alpha) \subseteq W_\alpha$, for all $\alpha \in G$. The category of G -graded vector spaces is denoted by \mathbf{Vect}^G .

A map $\varepsilon : G \times G \rightarrow \mathbb{K}^*$ is called a bicharacter on G if the following identities hold,

$$\begin{aligned}\varepsilon(\alpha, \beta + \gamma) &= \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma), \\ \varepsilon(\alpha + \beta, \gamma) &= \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \\ \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) &= 1,\end{aligned}$$

for all $\alpha, \beta, \gamma \in G$. We assume that G is a fixed abelian group and ε is a fixed bicharacter all though the paper.

The concept of Lie color algebras was introduced in [13] under the name of ε -Lie algebras.

Definition 2.1. A Lie color algebra or ε -Lie algebras is a G -graded vector space $L = \bigoplus_{\alpha \in G} L_\alpha$ together with a bracket $[\cdot, \cdot] : L \otimes L \rightarrow L$ satisfies,

- (i) graded condition: $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, $\forall \alpha, \beta \in G$
- (ii) ε -skew symmetry:

$$[x, y] + \varepsilon(x, y)[y, x] = 0, \quad (1)$$

- (iii) ε -Jacobi identity:

$$J_1(x, y, z) := \varepsilon(z, x)[[x, y], z] + \varepsilon(x, y)[[y, z], x] + \varepsilon(y, z)[[z, x], y] = 0. \quad (2)$$

for all homogeneous elements $x \in L_\alpha, y \in L_\beta, z \in L_\gamma$.

In the above definition, we write $\varepsilon(x, y)$ instead of $\varepsilon(|x|, |y|) = \varepsilon(\alpha, \beta)$ as in [11]. In fact, the ε -Jacobi identity can be write in another forms:

$$J_2(x, y, z) := [[x, y], z] - [x, [y, z]] + \varepsilon(x, y)[y, [x, z]] = 0. \quad (3)$$

Example 2.2. Let $A = \bigoplus_{\alpha \in G} A_\alpha$ be a G -graded associative algebra with multiplication $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. Define the bracket

$$[x, y] := xy - \varepsilon(x, y)yx,$$

for all homogeneous elements $x \in A_\alpha, y \in A_\beta$. Then $(A, [\cdot, \cdot])$ is a Lie color algebra.

Let G be an abelian group and $V = \bigoplus_{\alpha \in G} V_\alpha$ an G -graded vector space. Then the associative algebra $\text{End } V$ is equipped with the induced G -grading $\text{End}(V) = \bigoplus_{\alpha \in G} \text{End}(V)_\alpha$, where

$$\text{End}(V)_\alpha = \{A \in \text{End}(V) | A(V_\alpha) \subseteq V_{\gamma+\alpha}\}.$$

By Example 2.2 we get a Lie color algebra on $(\text{End}(V), [\cdot, \cdot])$, which is denoted by $\mathfrak{gl}(V)$.

A homomorphism between two Lie color algebras $(L, [\cdot, \cdot])$ and $(L', [\cdot, \cdot]')$ is a grade-perserving linear map $\varphi : L \rightarrow L'$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]'$$

for every $x, y \in L$.

Let L be a Lie color algebra and V is a G -graded vector space, a representation or an action of L on V is a homomorphism $\rho : L \rightarrow \mathfrak{gl}(V)$, i.e.

$$\rho([x, y])v = \rho(x)(\rho(y)v) - \varepsilon(x, y)\rho(y)(\rho(x)v), \quad (4)$$

for all $v \in V, x \in L_\alpha, y \in L_\beta$. We write $x \triangleright v = \rho(x)v$ to denote such an action. A new Lie color algebra can constructed as follows.

Proposition 2.3. *Let \mathfrak{g} be a Lie color algebra with bracket $[\cdot, \cdot]_{\mathfrak{g}}$, ρ is a representation of \mathfrak{g} on a G -graded vector space V . Define a ε -skew-symmetric bracket $[\cdot, \cdot]$ on $\mathfrak{g} \oplus V$ by*

$$[x + u, y + v] := [x, y]_{\mathfrak{g}} + x \triangleright v - \varepsilon(x, y)y \triangleright u. \quad (5)$$

Then $(\mathfrak{g} \oplus V, [\cdot, \cdot])$ becomes a Lie color algebra, which is called semidirect product of \mathfrak{g} and V .

In [10], Loday introduced a new algebraic structure, which is usually called Leibniz algebra. It's color version is as follows.

Definition 2.4. *An Leibniz color algebra is a G -graded vector space $L = \bigoplus_{\alpha \in G} L_{\alpha}$ together with a bracket $\circ : L \times L \rightarrow L$ satisfies $L_{\alpha} \circ L_{\beta} \subseteq L_{\alpha+\beta}$, $\forall \alpha, \beta \in G$, and the ε -Leibniz rule:*

$$x \circ (y \circ z) = (x \circ y) \circ z + \varepsilon(x, y) y \circ (x \circ z), \quad (6)$$

for all homogeneous elements $x \in L_{\alpha}, y \in L_{\beta}, z \in L_{\gamma}$.

3 Omni-Lie Color Algebras

Let V be a G -graded vector space. Like in [17], we define an operation \circ on $\mathcal{E} := \mathfrak{gl}(V) \oplus V$ by

$$(A + x) \circ (B + y) = [A, B] + Ay, \quad (7)$$

then we have

Proposition 3.1. *(\mathcal{E}, \circ) is a Leibniz color algebra.*

Proof. We have to check the ε -Leibniz rule for the operation \circ . Let $e_1 = A + x, e_2 = B + y, e_3 = C + z$, by definition,

$$\begin{aligned} & \{e_1 \circ e_2\} \circ e_3 - e_1 \circ \{e_2 \circ e_3\} - \varepsilon(x, y)e_2 \circ \{e_1 \circ e_3\} \\ &= ([A, B] + Ay) \circ (C + z) - (A + x) \circ ([B, C] + Bz) \\ & \quad - \varepsilon(x, y)(B + y) \circ ([A, C] + Az) \\ &= [[A, B], C] - [A, [B, C]] - \varepsilon(x, y)[B, [A, C]] \\ & \quad + [A, B]z - ABz - \varepsilon(x, y)BAz \\ &= 0 \end{aligned}$$

The right hand side of the above equation is zero because $\mathfrak{gl}(V)$ is a Lie color algebra acting on V . \square

Note that the above operation is not skew-symmetric, we can define a skew-symmetric bracket on $\mathcal{E} = \mathfrak{gl}(V) \oplus V$ as its skew symmetrization:

$$\llbracket A + x, B + y \rrbracket = [A, B] + \frac{1}{2} (Ay - \varepsilon(x, y)Bx), \quad (8)$$

and a symmetric bilinear form on \mathcal{E} with values in V :

$$\langle A + x, B + y \rangle = \frac{1}{2} (Ay + \varepsilon(x, y)Bx), \quad (9)$$

We call the triple $(\mathcal{E}, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)$ an **omni-Lie color algebra**.

Without the factor of $1/2$, this would be the semidirect product Lie color algebra for the action of $\mathfrak{gl}(V)$ on V . With the factor of $1/2$, the bracket does not satisfy the ε -Jacobi identity. This lead to the concept of 2-term color L_∞ -algebras defined in the next section. Now we compute the Jacobiator for this bracket.

Proposition 3.2. *For $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \mathcal{E}$, define*

$$\begin{aligned} T(e_1, e_2, e_3) := & \frac{1}{3} \{ \varepsilon(z, x) \langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + \varepsilon(x, y) \langle \llbracket e_2, e_3 \rrbracket, e_1 \rangle \\ & + \varepsilon(y, z) \langle \llbracket e_3, e_1 \rrbracket, e_2 \rangle \}. \end{aligned}$$

Let J_1 denote the Jacobiator given in (2) for the bracket $\llbracket \cdot, \cdot \rrbracket$ on \mathcal{E} , then we have

$$J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3).$$

Proof. We compute both the sides as follows:

$$\begin{aligned} & J_1(e_1, e_2, e_3) \\ = & \varepsilon(z, x) \llbracket \llbracket A + x, B + y \rrbracket, C + z \rrbracket + \text{c.p.} \\ = & \llbracket \varepsilon(z, x)[A, B] + \frac{1}{2} \varepsilon(z, x) (Ay - \varepsilon(x, y)Bx), C + z \rrbracket + \text{c.p.} \\ = & \varepsilon(z, x) \llbracket [A, B], C \rrbracket + \text{c.p.} \\ & + \frac{1}{2} (\varepsilon(z, x)[A, B]z - \frac{1}{2} \varepsilon(z, x) \varepsilon(x + y, z) C (Ay - \varepsilon(x, y)Bx)) \\ & + \frac{1}{2} (\varepsilon(x, y)[B, C]x - \frac{1}{2} \varepsilon(x, y) \varepsilon(y + z, x) A (Bz - \varepsilon(y, z)Cy)) \\ & + \frac{1}{2} (\varepsilon(y, z)[C, A]y - \frac{1}{2} \varepsilon(z, y) \varepsilon(z + x, y) B (Cx - \varepsilon(z, x)Az)) \\ = & \frac{1}{4} \varepsilon(z, x) ABz - \frac{1}{4} \varepsilon(z, x) \varepsilon(x, y) BAz + \frac{1}{4} \varepsilon(y, z) CAy \\ & - \frac{1}{4} \varepsilon(y, z) \varepsilon(x, y) CBx + \frac{1}{4} \varepsilon(x, y) BCx - \frac{1}{4} \varepsilon(z, x) \varepsilon(y, z) ACy, \end{aligned}$$

and

$$T(e_1, e_2, e_3)$$

$$\begin{aligned}
&= \frac{1}{3}\varepsilon(z, x)\langle [A + x, B + y], C + z \rangle + \text{c.p.} \\
&= \frac{1}{3}\varepsilon(z, x)\langle [A, B] + \frac{1}{2}(Ay - \varepsilon(x, y)Bx), C + z \rangle + \text{c.p.} \\
&= \frac{1}{6}\varepsilon(z, x)\left([A, B]z + \frac{1}{2}\varepsilon(x + y, z)C(Ay - \varepsilon(x, y)Bx)\right) + \text{c.p.} \\
&= \frac{1}{6}\varepsilon(z, x)ABz - \frac{1}{6}\varepsilon(z, x)\varepsilon(x, y)BAz + \frac{1}{12}\varepsilon(y, z)CAy - \frac{1}{12}\varepsilon(y, z)\varepsilon(x, y)CBx \\
&\quad + \frac{1}{6}\varepsilon(x, y)BCx - \frac{1}{6}\varepsilon(x, y)\varepsilon(y, z)CBx + \frac{1}{12}\varepsilon(z, x)ABz - \frac{1}{12}\varepsilon(z, x)\varepsilon(y, z)ACy \\
&\quad + \frac{1}{6}\varepsilon(y, z)CAy - \frac{1}{6}\varepsilon(y, z)\varepsilon(z, x)ACy + \frac{1}{12}\varepsilon(x, y)BCx - \frac{1}{12}\varepsilon(x, y)\varepsilon(z, x)BAz \\
&= \frac{1}{4}\varepsilon(z, x)ABz - \frac{1}{4}\varepsilon(z, x)\varepsilon(x, y)BAz + \frac{1}{4}\varepsilon(y, z)CAy \\
&\quad - \frac{1}{4}\varepsilon(y, z)\varepsilon(x, y)CBx + \frac{1}{4}\varepsilon(x, y)BCx - \frac{1}{4}\varepsilon(z, x)\varepsilon(y, z)ACy.
\end{aligned}$$

Thus, the two sides are equal. \square

The bracket $\llbracket \cdot, \cdot \rrbracket$ does not satisfy the Jacobi identity so that an omni-Lie color algebra is not a Lie color algebra. However, all possible Lie color algebra structures on V can be characterized by means of the omni-Lie color algebra.

For a graded operation $\omega : V_\alpha \times V_\beta \rightarrow V_{\alpha+\beta}$, we define the adjoint operator

$$\text{ad}_\omega : V_\alpha \rightarrow \mathfrak{gl}(V)_\alpha, \quad \text{ad}_\omega(x)(y) = \omega(x, y) \in V_{\alpha+\beta}$$

where $x \in V_\alpha, y \in V_\beta$. Then the graph of the adjoint operator:

$$\mathcal{F}_\omega = \{\text{ad}_\omega x + x ; \forall x \in V\} \subset \mathcal{E} = \mathfrak{gl}(V) \oplus V$$

is a subspace of \mathcal{E} . Denote \mathcal{F}_ω^\perp the orthogonal complement of \mathcal{F}_ω in \mathcal{E} with respect to the ε -symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{E} given in (9).

Proposition 3.3. *With the above notations, (V, ω) is a Lie color algebra if and only if its graph \mathcal{F}_ω is maximal isotropic, i.e. $\mathcal{F}_\omega = \mathcal{F}_\omega^\perp$, and is closed with respect to the bracket $\llbracket \cdot, \cdot \rrbracket$.*

Proof. If ω is ε -skew symmetric, i.e. $\omega(x, y) + \varepsilon(x, y)\omega(y, x) = 0$, then

$$\begin{aligned}
\langle \text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \rangle &= \frac{1}{2}(\text{ad}_\omega(x)y + \varepsilon(x, y)\text{ad}_\omega(y)x) \\
&= \frac{1}{2}(\omega(x, y) + \varepsilon(x, y)\omega(y, x))
\end{aligned}$$

This means that ω is ε -skew-symmetric if and only if its graph is isotropic, i.e. $\mathcal{F}_\omega \subseteq \mathcal{F}_\omega^\perp$. Moreover, by dimension analysis, we have \mathcal{F}_ω is maximal isotropic.

Next let $[x, y] := \omega(x, y)$, we shall check that the ε -Jacobi identity on V is satisfied if and only if \mathcal{F}_ω is closed under bracket (8) on \mathcal{E} . In fact,

$$\llbracket \text{ad}_\omega(x) + x, \text{ad}_\omega(x) + y \rrbracket$$

$$\begin{aligned}
&= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\text{ad}_\omega(x)y - \varepsilon(x, y) \text{ad}_\omega(y)x) \\
&= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\omega(x, y) - \varepsilon(x, y)\omega(y, x)) \\
&= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \omega(x, y).
\end{aligned}$$

Thus this bracket is closed if and only if

$$[\text{ad}_\omega(x), \text{ad}_\omega(y)] = \text{ad}_\omega(\omega(x, y)).$$

In this case, for $\forall z \in V$, we have

$$\begin{aligned}
&[\text{ad}_\omega(x), \text{ad}_\omega(y)](z) - \text{ad}_\omega(\omega(x, y))(z) \\
&= \text{ad}_\omega(x) \text{ad}_\omega(y)(z) - \varepsilon(x, y) \text{ad}_\omega(y) \text{ad}_\omega(x)(z) - \text{ad}_\omega(\omega(x, y))(z) \\
&= \text{ad}_\omega(x)\omega(y, z) - \varepsilon(x, y) \text{ad}_\omega(y)\omega(x, z) - \omega(\omega(x, y), z) \\
&= \omega(x, \omega(y, z)) - \varepsilon(x, y)\omega(y, \omega(x, z)) - \omega(\omega(x, y), z) \\
&= [x, [y, z]] - \varepsilon(x, y)[y, [x, z]] - [[x, y], z] \\
&= 0.
\end{aligned}$$

This is exactly the ε -Jacobi identity on V . \square

We define a Dirac structure of $\mathfrak{gl}(V) \oplus V$ to be any maximal isotropic subspace $L \subseteq \mathfrak{gl}(V) \oplus V$ which is closed under the bracket operation, then we have (V, ω) is a Lie color algebra if and only if \mathcal{F}_ω is a Dirac structure of the omni-Lie color algebra $\mathfrak{gl}(V) \oplus V$.

According to Proposition 3.2, for a Dirac structure L , we have

$$J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3) = 0, \quad \forall e_i \in L.$$

Thus a Dirac structure is a Lie color algebra, though omni-Lie color algebra is not for itself.

For a general characterization for all Dirac structures of \mathcal{E} , we adapt the theory of characteristic pairs developed in [8] (see also [15]).

For a maximal isotropic subspace $L \subset \mathfrak{gl}(V) \oplus V$, set the subspace $D = L \cap \mathfrak{gl}(V)$. Define $D^0 \subset V$ to be the null space of D :

$$D^0 = \{x \in V \mid X(x) = 0, \forall X \in D\}.$$

It is easy to see that $D = (D^0)^0$.

Lemma 3.4. *With notations above, a subspace L is maximal isotropic if and only if L is of the form*

$$L = D \oplus \mathcal{F}_{\pi|_{D^0}} = \{X + \pi(x) + x \mid X \in D, x \in D^0\}, \quad (10)$$

where $\pi : V \rightarrow \mathfrak{gl}(V)$ is a ε -skew-symmetric map.

Proof. In the following, we also denote $\pi(x, y) = \pi(x)(y) \in V$ for convenience. First suppose that L is given by (10), then

$$\begin{aligned} & \langle X + \pi(x) + x, Y + \pi(y) + y \rangle \\ &= \frac{1}{2} \{ X(y) + \pi(x, y) + \varepsilon(x, y)Y(x) + \varepsilon(x, y)\pi(y, x) \} \\ &= \frac{1}{2} \{ \pi(x, y) + \varepsilon(x, y)\pi(y, x) \} \\ &= 0, \quad \forall X + \pi(x) + x, Y + \pi(y) + y \in L, \end{aligned}$$

since $\pi : V \rightarrow \mathfrak{gl}(V)$ is ε -skew-symmetric so that L is isotropic. Next we prove that L is maximal isotropic. For $\forall Z + z \in L^\perp$,

$$\langle X, Z + z \rangle = X(z) = 0, \quad \forall X \in D \Rightarrow z \in D^0.$$

Moreover, $\forall x \in D^0$, the equality below

$$\begin{aligned} \langle X + \pi(x) + x, C + z \rangle &= X(z) + \pi(x)(z) + \varepsilon(x, z)Cx \\ &= \varepsilon(x, z)(C - \pi(z))(x) = 0, \end{aligned}$$

implies that $C - \pi(z) \triangleq Z \in D$. Thus

$$C + z = Z + \pi(z) + z \in L = D \oplus \mathcal{F}_{\pi|_{D^0}} \Rightarrow L = L^\perp.$$

The converse part is straightforward so we omit the details. \square

Lemma 3.5. *Let (D, π) be given above. Then L is a Dirac structure if and only if the following conditions are satisfied:*

- (1) D is a subalgebra of $\mathfrak{gl}(V)$;
- (2) $\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \quad \forall x, y \in D^0$;
- (3) $\pi(x, y) \in D^0, \quad \forall x, y \in D^0$.

Such a pair (D, π) is called a **characteristic pair** of a Dirac structure L . The proof is skipped since it is straightforward and similar to that in [8].

By means of the two lemmas above, we have

Proposition 3.6. *There is a one-to-one correspondence between Dirac structures of the omni-Lie color algebra $(\mathfrak{gl}(V) \oplus V, \langle \cdot, \cdot \rangle, \llbracket \cdot, \cdot \rrbracket)$ and Lie color algebra structures on subspaces of V .*

Proof. For any Dirac structure $L = D \oplus \mathcal{F}_{\pi|_{D^0}}$, a Lie color algebra structure on D^0 is as follows:

$$[x, y]_{D^0} \triangleq \pi(x, y) \in D^0, \quad \forall x, y \in D^0.$$

It easy to see that this is a ε -skew-symmetric map. For ε -Jacobi identity, we have for all $x, y, z \in D^0$,

$$\begin{aligned} [[x, y]_{D^0}, z]_{D^0} &= \pi([x, y]_{D^0})(z) = \pi((\pi(x)(y))(z) = [\pi(x), \pi(y)](z)) \\ &= \pi(x)(\pi(y)(z)) - \varepsilon(x, y)\pi(y)(\pi(x)(z)) \\ &= [x, [y, z]_{D^0}]_{D^0} - \varepsilon(x, y)[y, [x, z]_{D^0}]_{D^0}. \end{aligned}$$

Thus we get a Lie color algebra $(D^0, [\cdot, \cdot]_{D^0})$.

Conversely, for any Lie color algebra $(W, [\cdot, \cdot]_W)$ on a subspace W of V . Define D by

$$D = W^0 \triangleq \{X \in \mathfrak{gl}(V) \mid X(x) = 0, \quad \forall x \in W\}.$$

Then $D^0 = (W^0)^0 = W$. Since Lie color algebra structure $[\cdot, \cdot]_W$ gives a ε -skew symmetric morphism:

$$\text{ad} : W \rightarrow \mathfrak{gl}(W), \quad \text{ad}_x(y) = [x, y]_W,$$

we take a ε -skew symmetric morphism $\pi : V \rightarrow \mathfrak{gl}(V)$, as an extension of ad from $W = D^0$ to V . Thus we get a maximal isotropic subspace $L = D \oplus \mathcal{F}_{\pi|_W}$ from the pair (D, π) as in Lemma 3.4.

We shall prove that L is a Dirac structure. Firstly, $\forall X, Y \in D$ and $x \in W$, we have

$$[X, Y](x) = XY(x) - \varepsilon(X, Y)YX(x) = 0,$$

which implies that D is a subalgebra of $\mathfrak{gl}(V)$.

Next step is to prove that L is closed under the bracket $[[\cdot, \cdot]]$. Remember that $\pi|_W = \text{ad}$ and $[\cdot, \cdot]_W$ satisfies the ε -Jacobi identity, we obtain

$$[\text{ad}_x, \text{ad}_y] = \text{ad}_{[x, y]_W} = \text{ad}_{\text{ad}_x y}, \quad \forall x, y \in W.$$

For any $X \in D$ and $x, y \in W$, we have

$$[X, \text{ad}_x](y) = X([x, y]_W) - \varepsilon(X, x)[x, X(y)] = 0,$$

thus $[X, \text{ad}_x] \in D$. On the other hand, we have

$$[[X + \text{ad}_x + x, Y + \text{ad}_y + y]]$$

$$\begin{aligned}
&= [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + [\text{ad}_x, \text{ad}_y] + \frac{1}{2}(\text{ad}_x(y) - \varepsilon(x, y) \text{ad}_y(x)) \\
&= [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + \text{ad}_{[x, y]_W} + [x, y]_W \\
&\in D \oplus \mathcal{F}_{\pi|_W},
\end{aligned}$$

Thus, we conclude that L is a Dirac structure. Finally, it is easy to see that the Dirac structure L is independent of the choice of extension π . This completes the proof. \square

Let V be a Lie color algebra, recall that a map $D \in \mathfrak{gl}(V)$ is called a homogeneous color derivation of degree $|D|$ if $D(V_\alpha) \subseteq V_{\alpha+|D|}$ and

$$D([x, y]) = [Dx, y] + \varepsilon(D, x)[x, Dy].$$

Denote by $\text{Der}(V) = \bigoplus_{\alpha \in G} \text{Der}(V)_\alpha$, where $\text{Der}(V)_\alpha$ is the vector space spanned by all homogeneous color derivation of degree α . We will proof that $\text{Der}(V)$ becomes a Lie color algebra under the bracket

$$[D, D'] = DD' - \varepsilon(D, D')D'D,$$

where D, D' are homogeneous color derivations of degree $|D|, |D'|$.

Proposition 3.7. *D is a homogeneous color derivation of V if and only if \mathcal{F}_ω is an invariant subspace of D under the bracket \circ , i.e. $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$.*

Proof. For $\text{ad}_\omega(x) + x \in \mathcal{F}_\omega$,

$$D \circ (\text{ad}_\omega(x) + x) = [D, \text{ad}_\omega(x)] + Dx.$$

The right hand side is belong to \mathcal{F}_ω if and only if

$$[D, \text{ad}_\omega(x)] = \text{ad}_\omega(Dx),$$

that is

$$\begin{aligned}
&D \text{ad}_\omega(x)(y) - \varepsilon(D, x) \text{ad}_\omega(x)D(y) - \text{ad}_\omega(Dx)(y) \\
&= D[x, y] - \varepsilon(D, x)[x, D(y)] - [Dx, y] \\
&= 0.
\end{aligned}$$

Thus D is a derivation if and only if $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$. \square

We call the set of elements $D \in \mathfrak{gl}(V)$ such that $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$ the **normalizer** of \mathcal{F}_ω .

Proposition 3.8. *Let D, D' are homogeneous color derivations, then $[D, D']$ is also a homogeneous color derivation. Thus we have $\text{Der}(V) = N(\mathcal{F}_\omega)$ is a Lie color subalgebra of $\mathfrak{gl}(V)$.*

Proof. Let $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$ and $D' \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$, then

$$[D, \text{ad}_\omega(x)] = \text{ad}_\omega(Dx), \quad [D', \text{ad}_\omega(x)] = \text{ad}_\omega(D'x).$$

By the ε -Jacobi identity on $\mathfrak{gl}(V)$, we have

$$\begin{aligned} & [[D, D'] \circ \text{ad}_\omega(x)] \\ &= [D, [D', \text{ad}_\omega(x)]] + \varepsilon(D', x)[[D, \text{ad}_\omega(x)], D'] \\ &= [D, \text{ad}_\omega(D'x)] + \varepsilon(D', x)\varepsilon(D + x, D')[D', \text{ad}_\omega(Dx)] \\ &= \text{ad}_\omega(DD'x) + \varepsilon(D, D')\text{ad}_\omega(D'Dx) \\ &= \text{ad}_\omega([D, D']x). \end{aligned}$$

This is equivalent to $[D, D'] \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$, so the bracket in $\text{Der}(V)$ is closed. Thus $\text{Der}(V)$ is a Lie color algebra as a Lie color subalgebra of $\mathfrak{gl}(V)$. \square

4 2-term color L_∞ -algebras

As we mentioned in above section, the bracket operation $\llbracket \cdot, \cdot \rrbracket$ does not satisfy the ε -Jacobi identity, but up to some homotopy items. This leads to the concept of 2-term color L_∞ -algebras.

Definition 4.1. A 2-term color L_∞ -algebra $V = V_0 \oplus V_1$ is a complex consisting of the following data:

- two G -graded vector spaces V_0 and V_1 together with a grade-preserving linear map $d: V_1 \rightarrow V_0$, $d((V_1)_\alpha) \subseteq (V_0)_\alpha$.
- a bilinear map $l_2 = [\cdot, \cdot]: V_i \times V_j \rightarrow V_{i+j}$, where $0 \leq i + j \leq 1$,
- a trilinear map $l_3: V_0 \times V_0 \times V_0 \rightarrow V_1$.

These maps satisfy:

- (a) $[x, y] + \varepsilon(x, y)[y, x] = 0$,
- (b) $[x, h] + \varepsilon(x, h)[h, x] = 0$,
- (c) $[h, k] = 0$,
- (d) $l_3(x, y, z)$ is totally ε -skew symmetric.
- (e) $d([x, h]) = [x, dh]$,
- (f) $[dh, k] = [h, dk]$,

$$\begin{aligned}
(g) \quad d(l_3(x, y, z)) &= -[[x, y], z] + [x, [y, z]] + \varepsilon(y, z)[[x, z], y], \\
(h) \quad l_3(x, y, dh) &= -[[x, y], h] + [x, [y, h]] + \varepsilon(y, h)[[x, z], h], \\
(i) \quad \delta l_3(x, y, z, w) &:= [x, l_3(y, z, w)] - \varepsilon(x, y)[y, l_3(x, z, w)] \\
&\quad + \varepsilon(y + z, z)[z, l_3(x, y, w)] - [l_3(x, y, z), w] - l_3([x, y], z, w) \\
&\quad + \varepsilon(y, z)l_3([x, z], y, w) - \varepsilon(y + z, w)l_3([x, w], y, z) \\
&\quad - l_3(x, [y, z], w) + \varepsilon(z, w)l_3(x, [y, w], z) - l_3(x, y, [z, w]) = 0.
\end{aligned}$$

for all homogeneous elements $x, y, z, w \in V_0$ and $h, k \in V_1$.

This is the color analogue of a 2-term L_∞ -algebra studied in [2]. Omni-Lie color algebras give examples of this kind of structure.

Now, for a G -graded vector space V , let

$$V_0 = \mathfrak{gl}(V) \oplus V, \quad V_1 = V, \quad d = i : V \hookrightarrow \mathfrak{gl}(V) \oplus V$$

where i is the inclusion map and define

$$l_2 = \llbracket \cdot, \cdot \rrbracket, \quad l_3 = -\varepsilon(z, x)T.$$

Theorem 4.2. *With notations above, the omni-Lie color algebra $(\mathcal{E}, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)$ defines a Lie color 2-algebra $(V \xrightarrow{d} \mathfrak{gl}(V) \oplus V, l_2, l_3)$.*

Proof. For condition (a), by the grading in $\mathfrak{gl}(V) \oplus V$ we have $\deg(A + x) = \deg(A) = \deg(x)$, then

$$\begin{aligned}
&\llbracket A + x, B + y \rrbracket + \varepsilon(x, y)\llbracket B + y, A + x \rrbracket \\
= & [A, B] + \frac{1}{2}(Ay - \varepsilon(x, y)Bx) + \varepsilon(A, B)[B, A] \\
& + \varepsilon(x, y)\frac{1}{2}(Bx - \varepsilon(y, x)Ay) \\
= & [A, B] + \varepsilon(x, y)[B, A] + \frac{1}{2}(Ay - \varepsilon(x, y)Bx) \\
& + \frac{1}{2}(\varepsilon(x, y)Bx - Ay) \\
= & 0.
\end{aligned}$$

Conditions (b), (c), (e) and (f) can be checked easily. By Proposition 3.2, $l_3 = J_2$, thus (g)–(h) hold.

We now in a position to check the condition (i). For the case $e_1 = A, e_2 = B, e_3 = C, e_4 = w$.

$$[A, l_3(B, C, w)] - \varepsilon(x, y)[B, l_3(A, C, w)]$$

$$\begin{aligned}
& +\varepsilon(x+y, z)[C, l_3(A, B, w)] - [l_3(A, B, C), w] - l_3([A, B], C, w) \\
& +\varepsilon(y, z)l_3([A, C], B, w) - \varepsilon(y+z, w)l_3([A, w], B, C) \\
& -l_3(A, [B, C], w) + \varepsilon(z, w)l_3(A, [B, w], C) - l_3(A, B, [C, w]) \\
= & -\frac{1}{8}A[B, C]w + \frac{1}{8}\varepsilon(x, y)B[A, C]w - \frac{1}{8}\varepsilon(x+y, z)C[A, B]w + 0 \\
& +\frac{1}{4}[[A, B], C]w - \frac{1}{4}\varepsilon(y, z)[[A, C], B]w + \frac{1}{4}\varepsilon(x, y)\varepsilon(x, z)[[B, C], A]w \\
& +\varepsilon(x, y+z)[B, C]Aw - \frac{1}{8}\varepsilon(y, z)[A, C]Bw + \frac{1}{8}[A, B]Cw \\
= & \frac{1}{4}\{[[A, B], C] + \varepsilon(x, y+z)[[B, C], A] + \varepsilon(x+y, z)[[C, A], B]\}w \\
& -\frac{1}{8}\{A[B, C] - \varepsilon(x, y)B[A, C] + \varepsilon(x+y, z)C[A, B] \\
& -\varepsilon(x, y+z)[B, C]A + \varepsilon(y, z)[A, C]B - [A, B]C\}w \\
= & \frac{1}{4}\{[[A, B], C] - [A, [B, C]] + \varepsilon(x, y)[B, [A, C]]\}w \\
& -\frac{1}{8}\{[A, [B, C]] - \varepsilon(x, y)[B, [A, C]] - [[A, B], C]\}w \\
= & 0.
\end{aligned}$$

The general case can be checked similarly. \square

A 2-term color L_∞ -algebra is called **skeletal** if $d = 0$. In this case, from Conditions (a) and (g), we have V_0 is an ordinary Lie color algebra. Conditions (b) and (h) imply that V_1 is a representation of V_0 by the action defined by $x \triangleright h := [x, h]$. Now condition (i) can be described in terms of Lie color algebra cohomology (see [14]) with values in V_1 .

Proposition 4.3. *Skeletal 2-term color L_∞ -algebras are in one-to-one correspondence with quadruples $(\mathfrak{g}, V, \rho, l_3)$ where \mathfrak{g} is a Lie color algebra, V is a G -graded vector space, ρ is a representation of \mathfrak{g} on V and l_3 is a 3-cocycle on \mathfrak{g} with values in V .*

Recall that a quadratic Lie color algebra (see [1, 18]) is Lie color algebra $(\mathfrak{g}, [\cdot, \cdot])$ together with a ε -symmetric, nondegenerate, invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$, such that for any $x, y, z \in \mathfrak{g}$,

$$B(x, y) = \varepsilon(x, y)B(y, x), \quad B([x, y], z) = B(x, [y, z]).$$

Example 4.4. *Given a quadratic Lie color algebra $(\mathfrak{g}, [\cdot, \cdot], B)$, we construct a 2-term color L_∞ -algebra as follows. Let $V_1 = \mathbb{R}, V_0 = \mathfrak{g}, d = 0$, then define l_2, l_3 by*

$$l_2(x, y) = [x, y], \quad l_2(x, h) = 0, \quad l_3(x, y, z) = B([x, y], z), \quad (11)$$

where $x, y, z \in \mathfrak{g}, h \in \mathbb{R}$. All the conditions in the Definition 4.1 are satisfied, and we get a 2-term color L_∞ -algebra $(\mathbb{R} \xrightarrow{0} \mathfrak{g}, l_2, l_3)$ from a quadratic Lie color algebra $(\mathfrak{g}, [\cdot, \cdot], B)$. We call this **string Lie color 2-algebra**.

Another kind of 2-term color L_∞ -algebra is called **strict** if $l_3 = 0$. This kind of Lie color 2-algebras can be described in terms of crossed modules.

Definition 4.5. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ are two Lie color algebras. A crossed module of Lie color algebras is a homomorphism of Lie color algebras $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$ together with an action of \mathfrak{g} on \mathfrak{h} (i.e. a map $\triangleright : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ denoted by $\triangleright(x, h) = x \triangleright h$) such that

$$\varphi(x \triangleright h) = [x, \varphi(h)]_{\mathfrak{g}}, \quad \varphi(h) \triangleright k = [h, k]_{\mathfrak{h}},$$

for all $h, k \in \mathfrak{h}, x \in \mathfrak{g}$.

Proposition 4.6. Strict 2-term color L_∞ -algebras are in one-to-one correspondence with crossed modules of Lie color algebras.

Proof. Let $V_1 \xrightarrow{d} V_0$ be a 2-term color L_∞ -algebra, define $\mathfrak{g} = V_0$, $\mathfrak{h} = V_1$, and the following two bracket operations on \mathfrak{g} and \mathfrak{h} :

$$\begin{aligned} [h, k]_{\mathfrak{h}} &:= l_2(dh, k) = [dh, k], \quad \forall x, y \in \mathfrak{h} = V_1; \\ [x, y]_{\mathfrak{g}} &:= l_2(x, y) = [x, y], \quad \forall h, k \in \mathfrak{g} = V_0. \end{aligned}$$

Let $\varphi = d$, by Condition (a) and (g) in Definition 4.1, it is easy to see that $[\cdot, \cdot]_{\mathfrak{g}}$ satisfies the ε -Jacobi identity. By (h), we have

$$\begin{aligned} & -[[h, k]_{\mathfrak{h}}, l]_{\mathfrak{h}} + \varepsilon(k, l)[[h, l]_{\mathfrak{h}} k]_{\mathfrak{h}} + [h, [k, l]_{\mathfrak{h}}]_{\mathfrak{h}} \\ &= -[d[dh, k], l] + \varepsilon(k, l)[d[dh, l], k] + [dh, [dk, l]] \\ &= -[[dh, dk], l] + \varepsilon(k, l)[[dh, dl], k] + [dh, [dk, l]] \\ &= 0. \end{aligned}$$

Thus $[\cdot, \cdot]_{\mathfrak{h}}$ satisfies the ε -Jacobi identity. By (e), we have

$$\varphi([h, k]_{\mathfrak{h}}) = d([dh, k]) = [dh, dk] = [\varphi(h), \varphi(k)]_{\mathfrak{g}},$$

which implies that φ is a homomorphism of Lie color algebras.

Now define the maps of $\triangleright : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ by

$$x \triangleright h := l_2(x, h) = [x, h] \in \mathfrak{h},$$

which is an action by the equality

$$\begin{aligned} & [x, y] \triangleright h - x \triangleright y \triangleright h + \varepsilon(x, y)y \triangleright x \triangleright h \\ &= [[x, y], h] - [x, [y, h]] + \varepsilon(x, y)[y, [x, h]] \end{aligned}$$

$$= 0.$$

Finally, it is easy to check that

$$\begin{aligned}\varphi(x \triangleright h) &= d([x, h]) = [x, dh] = [x, \varphi(h)]_{\mathfrak{g}} \\ \varphi(h) \triangleright k &= [dh, k] = [h, k]_{\mathfrak{h}}.\end{aligned}$$

Therefore, we obtain a crossed module of Lie color algebras.

Conversely, a crossed module of Lie color algebras gives rise to a 2-term color L_∞ -algebra with $d = \varphi$, $V_0 = \mathfrak{g}$ and $V_1 = \mathfrak{h}$, where the brackets are given by:

$$\begin{aligned}l_2(x, y) &:= [x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}; \\ l_2(x, h) &:= x \triangleright h, \quad \forall x \in \mathfrak{g}; \\ l_2(h, k) &:= 0.\end{aligned}$$

The crossed module conditions give various conditions for 2-term color L_∞ -algebras with $l_3 = 0$. \square

Example 4.7. Let \mathfrak{g} be a Lie color algebra, $\text{Der}(\mathfrak{g})$ and $\text{Inn}(\mathfrak{g})$ be their derivations and inner derivations, then we have a crossed module $i : \text{Inn}(\mathfrak{g}) \rightarrow \text{Der}(\mathfrak{g})$, with $\text{Der}(\mathfrak{g})$ acting $\text{Inn}(\mathfrak{g})$ by $D \triangleright \text{ad}_x = \text{ad}_{Dx}$.

5 Lie Color 2-algebras

Recall that we denote the category of G -graded vector space by \mathbf{Vect}^G . We also call G -graded vector space by color vector space.

Definition 5.1. A color 2-vector space is a category in \mathbf{Vect}^G .

Thus, a color 2-vector space V is a category with a G -graded vector space of objects $V_0 = \bigoplus_{\alpha \in G} (V_0)_\alpha$ and a G -graded vector space of morphisms $V_1 = \bigoplus_{\alpha \in G} (V_1)_\alpha$, such that the source and target maps $s, t : V_1 \rightarrow V_0$, the identity-assigning map $i : V_0 \rightarrow V_1$, and the composition map $\circ : V_1 \times_{V_0} V_1 \rightarrow V_1$ are all grade-preserving linear maps. As in the ordinary case in [2], we write a morphism f from source x to target y by $f : x \rightarrow y$, i.e. $s(f) = x$ and $t(f) = y$. We also write $i(x)$ as 1_x .

Color 2-vector spaces are in one-to-one correspondence with 2-term complexes of color vector spaces. A 2-term complex of color vector spaces is a pair of color vector space with a differential between them: $C_1 \xrightarrow{d} C_0$.

color vector spaces. Roughly speaking, given a color 2-vector space V , $\text{Ker}(s) \xrightarrow{t} V_0$ is a 2-term complex. Conversely, any 2-term complex of color vector spaces $V_1 \xrightarrow{d} V_0$ gives rise to a color 2-vector space of which the set of objects is C_0 , the set of morphisms is $C_0 \oplus C_1$, the source map s is given by $s(x, h) = x$, and the target map t is given by $t(x, h) = x + dh$, where $x \in V_0$, $h \in V_1$. We denote the color 2-vector space associated to the 2-term complex of color vector spaces $V_1 \xrightarrow{d} V_0$ by V :

$$V = \begin{array}{ccc} V_1 & := & C_0 \oplus C_1 \\ & \downarrow s & \downarrow t \\ V_0 & := & C_0. \end{array} \quad (12)$$

Definition 5.2. A **Lie color 2-algebra** consists of a color 2-vector space L equipped with

- a ε -skew-symmetric bilinear functor, the **bracket**, $[\cdot, \cdot]: L \times L \rightarrow L$
- a completely ε -skew symmetric trilinear natural isomorphism, the **Jacobiator**,

$$J_{x,y,z}: [[x, y], z] \rightarrow [x, [y, z]] + \varepsilon(y, z)[[x, z], y],$$

such that the following Jacobiator identity is satisfied

$$\begin{aligned} & J_{[w,x],y,z}(1 + \varepsilon(y, z)[J_{w,x,z}, y])(J_{w,x,[y,z]} + \varepsilon(y, z)J_{w,[x,z],y} + \varepsilon(x + y, z)J_{[w,z],x,y}) \\ &= [J_{w,x,y}, z](J_{w,[x,y],z} + \varepsilon(x, y)J_{[w,y],x,z})([w, J_{x,y,z}] + 1 + 1 + \varepsilon(x, y + z)[J_{w,y,z}, x]) \end{aligned} \quad (13)$$

for all $w, x, y, z \in L_0$.

The Jacobiator identity looks similar as in [2], but they are not equal unless ε is trivial. When we draw it as a commutative diagram, we see that it relates two ways of using the Jacobiator to rebracket the expression

$[[[w, x], y], z]:$

$$\begin{array}{ccc}
& & [[w, x], y], z \\
& \swarrow J_{w, x, y, z} & \searrow J_{w, x, y, z} \\
[[w, [x, y]], z] + \varepsilon(x, y)[[w, y], x], z & & [[w, x], [y, z]] + \varepsilon(y, z)[[w, x], z], y \\
\downarrow J_{w, [x, y], z} + \varepsilon(x, y)J_{[w, y], x, z} & & \downarrow 1 + \varepsilon(y, z)J_{w, x, z, y} \\
\begin{aligned} & w, [[x, y], z] + \varepsilon(x + y, z)[[w, z], [x, y]] \\ & + \varepsilon(x, y)[[w, y], [x, z]] + \varepsilon(x, y)\varepsilon(x, z)[[[w, y], z], x] \end{aligned} & & \begin{aligned} & [[w, x], [y, z]] + \varepsilon(y, z)[[w, [x, z]], y] \\ & + \varepsilon(y, z)\varepsilon(x, z)[[[w, z], x], y] \end{aligned} \\
& \searrow [w, J_{x, y, z}] + 1 + 1 + \varepsilon(x, y + z)J_{w, y, z, x} & \swarrow J_{w, x, [y, z]} + \varepsilon(y, z)J_{w, [x, z], y} + \varepsilon(x + y, z)J_{[w, z], x, y} \\
& & P = Q
\end{array}$$

where P and Q are given by

$$\begin{aligned}
P &= [w, [x, [y, z]]] + \varepsilon(y, z)[w, [[x, z], y]] \\
&\quad + \varepsilon(x + y, z)[[w, z], [x, y]] + \varepsilon(x, y)[[w, y], [x, z]] \\
&\quad + \varepsilon(x, y)\varepsilon(x, z)[[w, [y, z]], x] + \varepsilon(x, y)\varepsilon(x, z)\varepsilon(y, z)[[[w, z], y], x], \\
Q &= [w, [x, [y, z]]] + \varepsilon(x, y + z)[[w, [y, z]], x] \\
&\quad + \varepsilon(y, z)[[w, [[x, z], y]] + \varepsilon(y, z)\varepsilon(x + z, y)[[w, y], [x, z]] \\
&\quad + \varepsilon(y, z)\varepsilon(x, z)[[w, z], [x, y]] + \varepsilon(y, z)\varepsilon(x, z)\varepsilon(x, y)[[[w, z], y], x].
\end{aligned}$$

In the case of ordinary Lie 2-algebras, it is well known that the category of ordinary Lie 2-algebras and the category of 2-term L_∞ -algebras are equivalent, see [2] for more details. Similarly, we have

Theorem 5.3. *There is a one-to-one correspondence between Lie color 2-algebras and 2-term color L_∞ -algebras.*

Proof. We give a sketch of the proof. We show how to construct a Lie color 2-algebra from a 2-term color L_∞ -algebra and how to construct a 2-term color L_∞ -algebra from a Lie color 2-algebra.

Let $V = (V_1 \xrightarrow{d} V_0, l_2, l_3)$ be a 2-term color L_∞ -algebra, we introduce a bilinear functor $[\cdot, \cdot]$ on the 2-vector space $L = (V_0 \oplus V_1 \rightrightarrows V_0)$ given by (12), that is L has color vector spaces of objects and morphisms $L_0 = V_0$, $L_1 = V_0 \oplus V_1$ and a morphisms $f: x \rightarrow y$ in L_1 by $f = (x, h)$ where homogenous elements $x \in V_0$ and $h \in V_1$ have the same degree. The source, target, and

identity-assigning maps in L are given by

$$\begin{aligned} s(f) &= s(x, h) = x, \\ t(f) &= t(x, h) = x + dh, \\ i(x) &= (x, 0), \end{aligned}$$

and we have $t(f) - s(f) = dh$.

Now we define bracket on L by

$$[(x, h), (y, k)] = l_2(x, y) + l_2(x, k) + l_2(h, y) + l_2(dh, k).$$

It is straightforward to see that it is a ε -skew-symmetric bilinear functor.

Next we define the Jacobiator as following

$$J_{x,y,z} := ([x, y], z], l_3(x, y, z)).$$

where $x, y, z \in V_0$ are homogenous elements.

Then by Condition (g), we have $J_{x,y,z}$ is a morphism from source $[[x, y], z]$ to target $[x, [y, z]] + \varepsilon(y, z)[[x, z], y]$.

Now we show that $J_{x,y,z}$ is natural isomorphism. We only check naturality in the third variable, the other two cases are similar. Let $f: z \rightarrow z'$. Then, $J_{x,y,z}$ is natural in z if the following diagram commutes:

$$\begin{array}{ccc} [[x, y], z] & \xrightarrow{[[1_x, 1_y], f]} & [[x, y], z'] \\ \downarrow J_{x,y,z} & & \downarrow J_{x,y,z'} \\ [x, [y, z]] + \varepsilon(y, z)[[x, z], y] & \xrightarrow{[1_x, [1_y, f]] + \varepsilon(y, z)[[1_x, f], 1_y]} & [x, [y, z']] + \varepsilon(y, z')[[x, z'], y] \end{array}$$

Using the formula for the composition and bracket in L this means that we need

$$([x, y], z], l_3(x, y, z') + [[x, y], h]) = ([x, y], z], \varepsilon(y, dh)[[x, h], y] + [x, [y, h]] + l_3(x, y, z)),$$

where $\deg(dh) = \deg(z)$ because $s(z, h) = z + dh \in V_0 \oplus V_1$. Thus, it suffices to show that

$$l_3(x, y, z') + [[x, y], h] = \varepsilon(y, dh)[[x, h], y] + [x, [y, h]] + l_3(x, y, z).$$

Then we have to show that:

$$l_3(x, y, z') + [[x, y], h] = l_3(x, y, z) + \varepsilon(y, dh)[[x, h], y] + [x, [y, h]],$$

or in other words,

$$[[x, y], h] + l_3(x, y, dh) = \varepsilon(y, dh)[[x, h], y] + [x, [y, h]].$$

This holds by condition (h) together with the complete antisymmetry of l_3 .

It is not hard to see that (13) is equivalent to

$$\begin{aligned} & l_3([w, x], y, z) + \varepsilon(y, z)l_2(l_3(w, x, z), y) + l_3(w, x, [y, z]) \\ & + \varepsilon(y, z)l_3(w, [x, z], y) + \varepsilon(x + y, z)l_3([w, z], x, y) \\ = & l_2(l_3(w, x, y), z) + l_3(w, [x, y], z) + \varepsilon(x, y)l_3([w, y], x, z) \\ & + l_2(w, l_3(x, y, z)) + \varepsilon(x, y + z)l_2(l_3(w, y, z), x). \end{aligned}$$

But this is condition (i) in Definition 4.1. Thus from a 2-term color L_∞ -algebra, we can obtain a Lie color 2-algebra.

Conversely, given a Lie color 2-algebra L , we define l_2 and l_3 on the 2-term complex $L_1 \supseteq \ker(s) = V_1 \xrightarrow{d} V_0 = L_0$ by

- $l_1 h = t(h)$ for $h \in V_1 \subseteq L_1$.
- $l_2(x, y) = [x, y]$ for $x, y \in V_0 = L_0$.
- $l_2(x, h) = [1_x, h]$ for $x \in V_0 = L_0$ and $h \in V_1 \subseteq L_1$.
- $l_2(h, k) = 0$ for $h, k \in V_1 \subseteq L_1$.
- $l_3(x, y, z) = p_1 J_{x, y, z}$ for $x, y, z \in V_0 = L_0$, where $p_1 : L_1 = V_0 \oplus V_1 \rightarrow V_1$ is the projection.

Then one can verify that $(V_1 \xrightarrow{d} V_0, l_2, l_3)$ is a 2-term L_∞ color algebra. \square

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Department of Mathematics and LMAM, Peking University, Beijing 100871, P. R. China;
E-mail address: `zhangtao@pku.edu.cn`.

College of Mathematics, Henan Normal University, Xinxiang 453007, P. R. China;
E-mail address: `zhangtao@htu.cn`